NON-NOETHERIAN COHEN-MACAULAY RINGS

TRACY DAWN HAMILTON AND THOMAS MARLEY

ABSTRACT. In this paper we investigate a property for commutative rings with identity which is possessed by every coherent regular ring and is equivalent to Cohen-Macaulay for Noetherian rings. We study the behavior of this property in the context of ring extensions (of various types) and rings of invariants.

1. INTRODUCTION

Over the past several decades Cohen-Macaulay rings have played a central role in the solutions to many important problems in commutative algebra and algebraic geometry. Hochster and Huneke [HH1] write that for many theorems "the Cohen-Macaulay condition (possibly on the local rings of a variety) is just what is needed to make the theory work." However, the study of this condition has mostly been restricted to the class of (commutative) Noetherian rings. (Of course, a non-Noetherian ring may be a Cohen-Macaulay module with respect to some Noetherian subring, but that is a separate – though not unrelated– issue.) The question has been raised by Glaz ([G2], [G4]) as to whether there exists a generalization of the Cohen-Macaulay property to non-Noetherian rings which has certain desirable properties. In particular, since many applications of Cohen-Macaulay rings use in an essential way that regular rings are Cohen-Macaulay, it is natural to look for a definition of Cohen-Macaulay for arbitrary rings which has the property that all (Noetherian and non-Noetherian) regular rings are Cohen-Macaulay. In [G4] Glaz asks whether such a definition exists, at least for coherent rings. We give an affirmative answer to this question.

The characterization of Noetherian Cohen-Macaulayness which we will extend is the following: A Noetherian ring R is Cohen-Macaulay if and only if any sequence $x_1, \ldots, x_n$ of elements of R generating a height n ideal is a regular sequence. This characterization will need some modification in the non-Noetherian case in order to have the desired properties. In particular, as heights of ideals can behave erratically in non-Noetherian rings, we replace the height condition on the sequence $\mathbf{x} = x_1, \ldots, x_n$ by conditions on the Čech cohomology $H^n_{\mathbf{x}}(R)$ and Koszul homologies $H^i(\mathbf{x}; R)$ for $i = 1, \ldots, n$. The conditions on the Koszul homologies – vacuous

in the Noetherian case – ensure that Cech cohomology is isomorphic to local coho-mology. The condition on Čech cohomology, namely that $H^n_{\mathbf{x}}(R)_p \neq 0$ for all primes p containing $(\mathbf{x})$, is equivalent to the condition $\mathrm{ht}(\mathbf{x}) = n$ in the Noetherian case but can be stronger if the ring is non-Noetherian. Sequences satisfying these con-ditions will be called *parameter sequences*. A parameter sequence such that every truncation (on the right) by any number of elements is also a parameter sequence is called a *strong parameter sequence*. We then call a ring *Cohen-Macaulay* if every strong parameter sequence is a regular sequence.

In section four of this paper we show that all coherent regular rings are Cohen-Macaulay under this definition (Theorem 4.8). We also give some results concerning the passage of the Cohen-Macaulay property along ring homomorphisms of special types (e.g., faithfully flat extensions, localizations, and quotients by a regular se-quence). While we are not able to give complete answers in all of these situations, it is clear that this notion of Cohen-Macaulay is less fluid for general rings than for Noetherian ones. For instance, we show that a quotient of a Cohen-Macaulay ring by a non-zero-divisor need not be Cohen-Macaulay. On the other hand, we prove a couple of results which demonstrate the utility of our definition. First, we show that if R is an excellent Noetherian domain of characteristic $p > 0$ then the integral closure R^+ of R in an algebraic closure of the fraction field of R is Cohen-Macaulay (Theorem 4.11), using the difficult result of [HH2] that R^+ is a big Cohen-Macaulay R-algebra. Secondly, we show that certain rings of invariants of coherent regular rings of dimension two are Cohen-Macaulay (cf. Corollary 4.15). This is a step toward resolving a conjecture posed by Glaz in [G2].

In section two we summarize the basic properties of Čech cohomology, weakly proregular sequences, and non-Noetherian grade. We establish a connection be-tween non-Noetherian grade and the vanishing of Čech cohomology which mirrors the situation for classical grade in Noetherian rings (Proposition 2.7). In section three we define parameter sequences and establish their basic properties.

Thoughout this article all rings are assumed to be commutative with identity and all modules unital. A ring with a unique maximal ideal is called 'quasi-local' while the term 'local' is reserved for Noetherian quasi-local rings. We let $\mathbf{x}$ denote a finite sequence of elements $x_1, \ldots, x_\ell$ of ring R. The length of the sequence $\mathbf{x}$ is denoted by $\ell(\mathbf{x})$. Given any sequence $\mathbf{x}$ of R we let $\mathbf{x}'$ denote the sequence obtained by truncating the last element from $\mathbf{x}$; i.e, $\mathbf{x}' = x_1, \ldots, x_{\ell-1}$ where $\ell = \ell(\mathbf{x})$.

2. Čech cohomology and non-Noetherian grade

Let R be a ring and x an element of R. Let $C(x)$ denote the cochain complex

$$0 \to R \xrightarrow[r \to \frac{r}{1}]{} R_x \to 0$$

where the position of R is in degree zero and the differential is the natural localiza-tion map. For a sequence $\mathbf{x}$ of elements of R, the Čech complex $C(\mathbf{x})$ is inductively defined by $C(\mathbf{x}) := C(\mathbf{x}') \otimes_R C(x_\ell)$, where $\ell = \ell(\mathbf{x})$. If M is an R-module, then we

set $C(\mathbf{x}; M) := C(\mathbf{x}) \otimes_R M$. The ith Čech cohomology $H^i_{\mathbf{x}}(M)$ of M with respect to the sequence $\mathbf{x}$ is defined to be the ith cohomology of $C(\mathbf{x}; M)$.

We list here some of the elementary properties of Čech cohomology. Proofs of these results are either elementary or can be found in section 5.1 of [BS].

Proposition 2.1. *Let R be a ring, $\mathbf{x}$ a finite sequence of elements of R and M an R-module.*

(a) $H^i_{\mathbf{x}}(M) = 0$ *for $i < 0$ and $i > \ell(\mathbf{x})$.*
(b) *Given a short exact sequence $0 \to A \to B \to C \to 0$ of R-modules there exists a natural long exact sequence*

$$\cdots \to H^i_{\mathbf{x}}(A) \to H^i_{\mathbf{x}}(B) \to H^i_{\mathbf{x}}(C) \to H^{i+1}_{\mathbf{x}}(A) \to \cdots.$$

(c) *There exists a long exact sequence*

$$\cdots \to H^i_{\mathbf{x}}(M) \to H^i_{\mathbf{x}'}(M) \xrightarrow{f_i} H^i_{\mathbf{x}'}(M)_{x_\ell} \to H^{i+1}_{\mathbf{x}}(M) \to \cdots$$

where f_i is (up to a sign) the natural localization map.
(d) *For all i the module $H^i_{\mathbf{x}}(M)$ is $(\mathbf{x})R$-torsion; i.e., every element is annihilated by a power of $(\mathbf{x})R$.*
(e) *If $\mathbf{y}$ is a finite sequence of elements of R such that $\sqrt{(\mathbf{x})R} = \sqrt{(\mathbf{y})R}$, then $H^i_{\mathbf{x}}(M) \cong H^i_{\mathbf{y}}(M)$ for all i.*
(f) *(Change of Rings) Let $f : R \to S$ be a ring homomorphism and N an S-module. Then $H^i_{\mathbf{x}}(N) \cong H^i_{f(\mathbf{x})}(N)$ for all i.*
(g) *(Flat Base Change) Let $f : R \to S$ be a flat ring homomorphism and M an R-module. Then $H^i_{\mathbf{x}}(M) \otimes_R S \cong H^i_{\mathbf{x}}(M \otimes_R S) \cong H^i_{f(\mathbf{x})}(M \otimes_R S)$ for all i.*
(h) $H^{\ell(\mathbf{x})}_{\mathbf{x}}(M) \cong H^{\ell(\mathbf{x})}_{\mathbf{x}}(R) \otimes_R M$ *and* $\operatorname{Supp}_R H^{\ell(\mathbf{x})}_{\mathbf{x}}(M) \subseteq \operatorname{Supp}_R M/(\mathbf{x})M$.

For an ideal I of R the ith *local cohomology* $H^i_I(M)$ of M with support in $V(I)$ is defined by

$$H^i_I(M) := \varinjlim \operatorname{Ext}^i_R(R/I^n, M).$$

In the case R is Noetherian we have $H^i_{\mathbf{x}}(M) \cong H^i_I(M)$ for all i, where $I = (\mathbf{x})R$. However, local cohomology and Čech cohomology are not in general isomorphic over non-Noetherian rings. Schenzel [Sch] gives necessary and sufficient conditions on a sequence $\mathbf{x}$ such that $H^i_{\mathbf{x}}(M) \cong H^i_I(M)$ for all i and R-modules M: For $x \in R$ let $K(x)$ denote the Koszul chain complex $0 \to R \xrightarrow{x} R \to 0$, where the first R is in degree 1. For a sequence $\mathbf{x} = x_1, \ldots, x_\ell$ the Koszul complex $K(\mathbf{x})$ is defined to be the chain complex $K(x_1) \otimes \cdots \otimes K(x_\ell)$. We denote the homology of $K(\mathbf{x})$ by $H_i(\mathbf{x})$. For $m \geq n$ there exists a chain map $\phi^m_n(\mathbf{x}) : K(\mathbf{x}^m) \to K(\mathbf{x}^n)$ given by $\phi^m_n(\mathbf{x}) = \phi^m_n(x_1) \otimes \cdots \otimes \phi^m_n(x_\ell)$, where $\phi^m_n(x)$ is the chain map

$$
\begin{array}{ccccccccc}
0 & \longrightarrow & R & \xrightarrow{x^m} & R & \longrightarrow & 0 \\
& & \downarrow{\scriptstyle x^{m-n}} & & \downarrow{\scriptstyle =} & & \\
0 & \longrightarrow & R & \xrightarrow{x^n} & R & \longrightarrow & 0.
\end{array}
$$

Hence, $\{K(\mathbf{x}^m), \phi_n^m(\mathbf{x})\}$ is an inverse system of chain complexes. The sequence $\mathbf{x}$ is called *weakly proregular* if for each n there exists an $m \geq n$ such that the maps $\phi_n^m(\mathbf{x})_* : H_i(\mathbf{x}^m) \to H_i(\mathbf{x}^n)$ are zero for all $i \geq 1$ [Sch]. Note that an element x is weakly proregular if and only if there exists an $n \geq 1$ such that $(0 : x^n) = (0 : x^{n+1})$. We note the following elementary remarks:

Remark 2.2. *Let R be a ring and $\mathbf{x}$ a finite sequence of elements from R.*

(a) If $\mathbf{x}$ is weakly proregular then so is any permutation of $\mathbf{x}$.
(b) Any regular sequence is proregular.
(c) Suppose $f : R \to S$ is a flat ring homomorphism. If $\mathbf{x}$ is weakly proregular on R then $f(\mathbf{x})$ is weakly proregular on S. The converse is true if f is faithfully flat.

We will need the following result, due to Schenzel:

Theorem 2.3. *[Sch, Theorem 3.2] Let R be a commutative ring, $\mathbf{x}$ a finite sequence of elements of R, and $I = (\mathbf{x})R$. The following conditions are equivalent:*

(a) $\mathbf{x}$ is weakly proregular.
(b) $H_{\mathbf{x}}^i(E) = 0$ for all injective R-modules E and $i \neq 0$.
(c) For every R-module M there exist natural isomorphisms $H_{\mathbf{x}}^i(M) \to H_I^i(M)$ for all i.

As a consequence of this theorem we obtain that any finite sequence of elements in a Noetherian ring is weakly proregular. Also, if $\sqrt{(\mathbf{x})R} = \sqrt{(\mathbf{y})R}$ where $\mathbf{x}$ and $\mathbf{y}$ are finite sequences (but not necessarily of the same length), then $\mathbf{x}$ is weakly proregular if and only if $\mathbf{y}$ is. As an example of an element in a ring which is not weakly proregular, consider the image of x in the ring $S = \mathbb{Z}[x, y_1, y_2, y_3, \dots]/(xy_1, x^2y_2, x^3y_3, \dots)$. By Schenzel's theorem, there exists an injective S-module E such that $H_x^1(E) \ncong H_{(x)}^0(E) = 0$.

Let R be a ring of Krull dimension d and $\mathbf{x}$ a sequence of elements of R. Two central results concerning Čech cohomology over a Noetherian ring are:

    (1) $H_{\mathbf{x}}^i(M) = 0$ for $i > d$.
    (2) If R is local and $\dim R/(\mathbf{x}) = 0$ then $H_{\mathbf{x}}^d(R) \neq 0$.

Part (2) may fail for non-Noetherian rings. For example, suppose V is a valuation domain of finite dimension $d > 1$. As the prime ideals of V are linearly ordered, there exists a non-unit $x \in V$ such that $\dim V/xV = 0$, while clearly $H_x^d(V) = 0$. On the other hand, (1) holds for arbitrary commutative rings:

Proposition 2.4. *Let R be a ring of finite dimension d and $\mathbf{x}$ a sequence of elements from R. Then $H_{\mathbf{x}}^i(M) = 0$ for all $i > d$.*

Proof: We use induction on d. It is enough to prove the statement in the case (R, m) is a quasi-local ring and $(\mathbf{x})R \subseteq m$. If $d = 0$ then every element of m is nilpotent. Hence, $H_{\mathbf{x}}^i(M) = 0$ for all $i \geq 1$. Now suppose that $d > 0$ and that the proposition holds for all rings of dimension at most $d - 1$. Let $\ell = \ell(\mathbf{x})$. For $\ell \leq d$ there is nothing to show. Suppose $\ell > d$ and let j be the largest integer such

that $H_{\mathbf{x}}^j(M) \neq 0$. Assume $j > d$. By induction on ℓ, $H_{\mathbf{x}'}^j(M) = 0$. From the exact sequence

$$\cdots \to H_{\mathbf{x}'}^{j-1}(M) \to H_{\mathbf{x}'}^{j-1}(M)_{x_\ell} \to H_{\mathbf{x}}^j(M) \to 0$$

we obtain that $H_{\mathbf{x}'}^{j-1}(M_{x_\ell}) \neq 0$. As R is quasi-local of dimension d, $\dim R_{x_\ell} \leq d-1$. Thus, $j - 1 \leq d - 1$, a contradiction. $\qquad\square$

We now briefly discuss non-Noetherian grade, also referred to as 'polynomial grade' [EN] and 'true grade' [No]. This notion dates back to the early 1970s (e.g., [Ba], [Ho]) in connection with the study of finite free resolutions over (arbitrary) commutative rings. Hochster appears to have been the first to notice that the pathological behavior of 'classical' grade in the non-Noetherian case can be remedied by adjoining indeterminates to the ring ([No, footnote, p.132]).

To begin we recall some terminology from [Ho]. Let R be a ring and M an R-module. A sequence $\mathbf{x} = x_1, \ldots x_\ell \in R$ is called a *possibly improper regular sequence* on M if x_i is a non-zero-divisor on $M/(x_1, \ldots, x_{i-1})M$ for $i = 1, \ldots, \ell$. If in addition $M \neq (\mathbf{x})M$ we call $\mathbf{x}$ a *regular sequence* on M. Given an ideal I of R, the *classical grade* of I on M, denoted $\mathrm{grade}(I, M)$, is defined to be the supremum of the lengths of all possibly improper regular sequences on M contained in I. In the case R is Noetherian and M is finitely generated, $\mathrm{grade}(I, M) > 0$ if and only if $(0 :_M I) = 0$. However, there are examples of finitely generated ideals in non-Noetherian rings which have annihilator zero but consist entirely of zero-divisors on the ring (cf. [V] or Example 2.10 below). This phenomenon disappears if one first passes to a polynomial ring extension of R. The following lemma is the central insight behind polynomial grade:

Lemma 2.5. *Let R be a ring, $I = (x_1, \ldots, x_\ell)R$, and M an R-module. Then $\mathrm{grade}(IR[t], R[t] \otimes_R M) > 0$ if and only if $(0 :_M I) = 0$. In particular, $(0 :_M I) = 0$ if and only if $x_1 + x_2 t + \cdots + x_\ell t^{\ell-1} \in IR[t]$ is a non-zero-divisor on $R[t] \otimes_R M$.*

Proof: See Chapter 5, Theorem 7 of [No]. $\qquad\square$

For an ideal I of R and R-module M, the *polynomial grade* of I on M is defined by

$$\text{p-grade}(I, M) := \lim_{m \to \infty} \mathrm{grade}(IR[t_1, \ldots, t_m], R[t_1, \ldots, t_m] \otimes_R M).$$

It is easily seen (cf. [A], [Ho]) that

$$\text{p-grade}(I, M) = \sup\{\mathrm{grade}(IS, S \otimes_R M) \mid S \text{ a faithfully flat } R\text{-algebra}\}.$$

We denote $\mathrm{grade}(I, R)$ and $\text{p-grade}(I, R)$ by $\mathrm{grade}\, I$ and $\text{p-grade}\, I$, respectively. If (R, m) is quasi-local, then $\mathrm{grade}(m, M)$ and $\text{p-grade}(m, M)$ are denoted by $\mathrm{depth}\, M$ and $\text{p-depth}\, M$, respectively. We note that if R is Noetherian then $\mathrm{grade}(I, M) = \text{p-grade}(I, M)$ for all ideals I and finitely generated R-modules M. The following proposition summarizes the essential properties of polynomial grade. Proofs of these results can be found in Chapter 5 of [No].

Proposition 2.6. *Let R be a ring and I an ideal of R.*

(a) If $\mathbf{x}$ is a regular sequence on M contained in I then

$$\text{p-grade}(I, M) = \text{p-grade}(I, M/(\mathbf{x})M) + \ell(\mathbf{x}).$$

(b) $\text{p-grade}(I, M) = \text{p-grade}(P, M)$ for some prime ideal P containing I. In particular, $\text{p-grade}(I, M) = \text{p-grade}(\sqrt{I}, M)$.

(c) $\text{p-grade}(I, M) = \sup\{\text{p-grade}(J, M) \mid J \subseteq I, \ J$ a finitely generated ideal$\}$.

(d) If I is generated by n elements and $IM \neq M$, then $\text{p-grade}\, I \leq n$. Furthermore, $\text{p-grade}(I, M) = \text{grade}(IR[t_1, \ldots, t_n], R[t_1, \ldots, t_n] \otimes_R M)$.

For a sequence $\mathbf{x}$ of R and an R-module M, the ith Koszul homology of $\mathbf{x}$ on M, denoted $H_i(\mathbf{x}; M)$, is defined to be the ith homology of $K(\mathbf{x}) \otimes_R M$. The following proposition relates polynomial grade with the vanishing of Koszul homology and Čech cohomology:

Proposition 2.7. *Let R be a ring, $\mathbf{x}$ a finite sequence of elements from R of length $\ell = \ell(\mathbf{x})$, $I = (\mathbf{x})R$, and M an R-module. The following integers (including the possibility of ∞) are equal:*

(1) $\text{p-grade}(I, M)$;

(2) $\sup\{k \geq 0 \mid H_{\ell-k}(\mathbf{x}, M) = 0 \ \text{ for all } i < k\}$;

(3) $\sup\{k \geq 0 \mid H_{\mathbf{x}}^i(M) = 0 \text{ for all } i < k\}$.

Moreover, $IM \neq M$ if and only if any one of the above integers is finite.

Proof: The equality of (1) and (2) was established by Barger [Ba] and Alfonsi [A]. We prove the equality of (1) and (3). Let $p = \text{p-grade}(I, M)$ and $h = h(\mathbf{x}, M)$ the quantity representing (3). We first assume $p < \infty$ and use induction on p to prove $p = h$. If $p = 0$ then by Lemma 2.5 we have $(0 :_M I) \neq 0$. Hence, $H_{\mathbf{x}}^0(M) \neq 0$ and $h = 0$. Assume now that $p > 0$. Then $\text{grade}(IR[t], R[t] \otimes_R M) > 0$. As $R \to R[t]$ is a faithfully flat ring extension, it follows by Proposition 2.1 that $H_{\mathbf{x}}^i(M) = 0$ if and only if $H_{\mathbf{x}}^i(R[t] \otimes_R M) = 0$ for all i. Hence, by replacing R by $R[t]$, we may assume $\text{grade}(I, M) > 0$. Let $u \in I$ be a non-zero-divisor on M. Since $\text{p-grade}(I, M/uM) = p - 1$, we have by induction that $h(\mathbf{x}, M/uM) = p - 1$. The short exact sequence $0 \to M \xrightarrow{u} M \to M/uM \to 0$ induces the long exact sequence

$$\cdots \to H_{\mathbf{x}}^{i-1}(M/uM) \to H_{\mathbf{x}}^i(M) \xrightarrow{u} H_{\mathbf{x}}^i(M) \to \cdots.$$

Thus, for $i \leq p-1$, multiplication by u on $H_{\mathbf{x}}^i(M)$ is injective. However, since $u \in I$ every element of $H_{\mathbf{x}}^i(M)$ is annihilated by a power of u. This implies $H_{\mathbf{x}}^i(M) = 0$ for $i \leq p - 1$. Finally, the same long exact sequence yields the exactness of

$$0 \to H_{\mathbf{x}}^{p-1}(M/uM) \to H_{\mathbf{x}}^p(M),$$

which implies $H_{\mathbf{x}}^p(M) \neq 0$. Thus, $h = p$.

Suppose $h < \infty$. As before, if $h = 0$ then $p = 0$. If $h > 0$ then by Lemma 2.5 we have $\text{grade}(IR[t], R[t] \otimes_R M) > 0$. Since $h(I, M) = h(IR[t], R[t] \otimes_R M)$, we may assume the existence of $u \in I$ which is a non-zero-divisor on M. Then $\text{p-grade}(I, M/uM) = p - 1$ and (using the same long exact sequence as above) $h(I, M/uM) = h - 1$. By induction, $h - 1 = p - 1$. $\square$

Let M be an R-module. A prime ideal P is said to be *weakly associated* to M if P is minimal over $(0 :_R x)$ for some $x \in M$ (cf. [Bk]). We denote the set of weakly associated primes of M by $\text{wAss}_R M$. It is easily seen that if R is Noetherian $\text{wAss}_R M = \text{Ass}_R M$ for all R-modules M. As in the Noetherian case, the union of the weakly associated primes of M is the set of zero-divisors on M and $\text{wAss}_R M = \emptyset$ if and only if $M = 0$. We prove the following elementary (and presumably well-known) result:

Lemma 2.8. *Let M be an R-module and $p \in \text{wAss}_R M$. Then $\text{p-depth}_{R_p} M_p = 0$.*

Proof: By localizing at p, we may assume (R, m) is quasi-local and $m = \sqrt{(0 :_R x)}$ for some $x \in M$. Let J be a finitely generated ideal contained in m. Then $J^n \subseteq (0 :_R x)$ for some n. Hence, $(0 :_M J^n) \neq 0$ which implies $\text{p-grade}(J, M) = \text{p-grade}(J^n, M) = 0$ by Lemma 2.5 and Proposition 2.6(b). By part (c) of Proposition 2.6, we obtain $\text{p-grade}(m, M) = 0$. $\square$

While polynomial grade has many of the same properties as classical grade for Noetherian rings, one important difference is that a ring may contain ideals of polynomial grade $j > 1$ but no ideals of polynomial grade i for $0 < i < j$. To see this, we first prove the following proposition, which is adapted from [V]:

Proposition 2.9. *Let (R, m) be a quasi-local ring of dimension d. Fix an integer $i \geq 0$ and let*

$$M_i := \bigoplus_{\substack{p \in \text{Spec } R \\ \text{ht } p \leq i}} k(p)$$

where $k(p)$ is the residue field of R_p. Let $S = R \times M_i$ be the trivial extension of R by M_i, $j : S \to R$ the natural projection, and I a finitely generated ideal of S. Then $\text{ht } I = \text{ht } j(I)$ and

$$\text{p-grade } I = \begin{cases} 0 & \text{if ht } I \leq i, \\ \text{p-grade } j(I) & \text{if ht } I > i. \end{cases}$$

Moreover, for any sequence $\mathbf{x}$ of S, $\mathbf{x}$ is weakly proregular on S if and only if $j(\mathbf{x})$ is weakly proregular on R.

Proof: Since the ideal $0 \times M$ is nilpotent, there is a bijective correspondence between $\text{Spec } S$ and $\text{Spec } R$ given by $P \to j(P)$. Hence, $\text{ht } I = \text{ht } j(I)$ and $\sqrt{I} = \sqrt{j(I)S}$. By Proposition 2.6(b), $\text{p-grade } I = \text{p-grade } j(I)S = \text{p-grade}(j(I), S)$. Thus, it suffices to show that for all finitely generated ideals J of R, $\text{p-grade}(J, S) = 0$ if $\text{ht } J \leq i$ and $\text{p-grade}(J, S) = \text{p-grade}(J, R)$ if $\text{ht } J > i$. Suppose first that $\text{ht } J \leq i$. Then J is contained in some prime p of R of height $j \leq i$. Let α be an element of M_i which is nonzero in the component corresponding to $k(p)$ and zero in all other components. Clearly, $Js = 0$ where $s = (0, \alpha) \in S$. Hence, $\text{p-grade}(J, S) = 0$ by Lemma 2.5.

Suppose that $\text{ht } J > i$. Let $J = (\mathbf{x})R$. For any prime p of height less than $\text{ht } J$ we have $(\mathbf{x})k(p) = k(p)$. By the change of rings isomorphism (Proposition 2.1(f)), $H_{\mathbf{x}}^i(k(p)) = 0$ for all $i \geq 0$. Therefore, $H_{\mathbf{x}}^i(M_i) = 0$ for all i. As $S \cong R \oplus M_i$ as

R-modules, we have $H^i_{\mathbf{x}}(S) \cong H^i_{\mathbf{x}}(R)$ for all i. Thus, p-grade$(J, S) =$ p-grade(J, R) by Proposition 2.7.

To prove the last statement, note that as $\sqrt{(\mathbf{x})S} = \sqrt{j(\mathbf{x})S}$, we have by Proposition 2.3 that $\mathbf{x}$ is weakly proregular on S if and only if $j(\mathbf{x})$ is weakly proregular on S. Thus, it suffices to prove that if $\mathbf{x}$ is a finite sequence of elements from R, then $\mathbf{x}$ is weakly proregular on R if and only if it is weakly proregular on S. However, as R-modules $H_j(\mathbf{x}^n; S) \cong H_j(\mathbf{x}^n; R) \oplus H_j(\mathbf{x}^n; M_i)$ for all j. Since M_i is a direct sum of fields, it is easy to see that the maps $H_j(\mathbf{x}^{n+1}; M_i) \to H_j(\mathbf{x}^n; M_i)$ are zero for all $j \geq 1$ and $n \geq 0$. (Note that if F is a field and $\mathbf{y}$ a sequence in F, then $H_i(\mathbf{y}, F) \neq 0$ for some i if and only if $\mathbf{y}$ is the zero sequence.) Hence, $\mathbf{x}$ is weakly proregular on R if and only if it is weakly proregular on S. $\qquad\square$

Applying this Proposition in the case R is a Cohen-Macaulay local ring, we get the following:

Example 2.10. *Let (R, m) be a Cohen-Macaulay local ring of dimension $d > 0$. Let $S = R \times M_{d-1}$ as in Proposition 2.9. Then S is a quasi-local ring of dimension d with maximal ideal $n = m \times M_{d-1}$ with the following properties:*
(a) p-depth $S = \dim S$.
(b) p-grade $I = 0$ for all ideals I of S such that $\sqrt{I} \neq n$; in particular, n consists entirely of zero-divisors.
(c) p-depth $S_P = 0$ for all $P \in \operatorname{Spec} S \setminus \{n\}$.

Proof: As we noted in the proof of Proposition 2.9, $\sqrt{I} = \sqrt{j(I)S}$ for every ideal I of S. As $j(I)$ is finitely generated (since R is Noetherian), we see that Proposition 2.9 applies to all ideals of S. Parts (a) and (b) now follow. For part (c), note that for any $P \in \operatorname{Spec} S$, $S_P \cong R_{j(P)} \times (M_{d-1})_{j(P)}$. $\qquad\square$

We end this section with a statement of the Auslander-Buchsbaum theorem for quasi-local rings. We denote the projective dimension of an R-module M by $\operatorname{pd}_R M$. A *finite free resolution* (FFR) of M is a resolution of M of finite length consisting of finitely generated free modules in each degree.

Proposition 2.11. *Let (R, m) be a quasi-local ring and M an R-module which has an FFR. Then*
$$\operatorname{pd}_R M + \text{p-depth}\, M = \text{p-depth}\, R.$$

Proof: See Chapter 6, Theorem 2 of [No]. $\qquad\square$

3. Parameter sequences

A sequence of elements $\mathbf{x}$ in a (Noetherian) local ring R is said to be a system of parameters (s.o.p.) if $\operatorname{ht}(\mathbf{x})R = \ell(\mathbf{x}) = \dim R$. If $\operatorname{ht}(\mathbf{x})R = \ell(\mathbf{x}) < \dim R$, we say $\mathbf{x}$ is a partial s.o.p. We wish to extend this notion to sequences in non-Noetherian rings using homological properties of the ring instead of height conditions.

Definition 3.1. Let R be a ring and M an R-module. A finite sequence $\mathbf{x}$ of elements of R is called a *parameter sequence* on R provided the following conditions hold:

(1) $\mathbf{x}$ is a weakly proregular sequence;

(2) $(\mathbf{x})R \neq R$;

(3) $H_{\mathbf{x}}^{\ell(\mathbf{x})}(R)_p \neq 0$ for all prime ideals p containing $(\mathbf{x})R$.

A parameter sequence of length one on R is called a *parameter* of R. The sequence $\mathbf{x}$ is called a *strong parameter sequence* on R if $x_1, \ldots, x_i$ is a parameter sequence on R for $i = 1, \ldots, \ell(\mathbf{x})$.

We define the ideal generated by the empty sequence to be the zero ideal. Thus, the empty sequence is a parameter sequence of length zero on any ring. The empty sequence will also be considered as a regular sequence of length zero on any ring. These conventions will allow us to begin proofs by induction with $\ell(\mathbf{x}) = 0$.

The following remark shows that parameter sequences coincide with (partial) systems of parameters if the ring is Noetherian.

Remark 3.2. *Let R be a Noetherian ring and $\mathbf{x}$ a finite sequence of elements of R. Then $\mathbf{x}$ is a parameter sequence on R if and only if $\mathrm{ht}(\mathbf{x})R = \ell(\mathbf{x})$.*

Proof: Recall that any sequence of elements in a Noetherian ring is a weakly proregular sequence. Also, by convention $\mathrm{ht}(\mathbf{x}) = \infty$ if and only if $(\mathbf{x})R = R$. Let p be a prime of height h which is minimal over $(\mathbf{x})$ and let $\ell = \ell(\mathbf{x})$. By Krull's Principle Ideal Theorem we have $h \leq \ell$. Since $(\mathbf{x})R_p$ is primary to pR_p and using standard facts about local cohomology, we get that $H_{\mathbf{x}}^{\ell}(R)_p \cong H_{pR_p}^{\ell}(R_p) \neq 0$ if and only if $h = \ell$. $\qquad\square$

We cite some elementary properties of parameter sequences:

Proposition 3.3. *Let R be a ring and $\mathbf{x}$ a finite sequence of elements of R.*

(a) Any permutation of a parameter sequence on R is again a parameter sequence on R.

(b) If $\sqrt{(\mathbf{y})R} = \sqrt{(\mathbf{x})R}$ and $\ell(\mathbf{y}) = \ell(\mathbf{x})$ then $\mathbf{x}$ is a parameter sequence on R if and only if $\mathbf{y}$ is a parameter sequence on R.

(c) Let $f : R \to S$ be a flat ring homomorphism. If $\mathbf{x}$ is a (strong) parameter sequence on R and $S/(f(\mathbf{x}))S \neq 0$ then $f(\mathbf{x})$ is a (strong) parameter sequence on S. The converse holds if f is faithfully flat.

(d) Let $f : R \to S$ be a ring homomorphism and $\mathbf{x}$ a weakly proregular sequence on R such that $(f(\mathbf{x}))S_p \neq S_p$ for all primes p of R minimal over $(\mathbf{x})R$. If $f(\mathbf{x})$ is a parameter sequence on S then $\mathbf{x}$ is a parameter sequence on R.

(e) If $\mathrm{p\text{-}grade}((\mathbf{x}), R) = \ell(\mathbf{x})$ then $\mathbf{x}$ is a parameter sequence on R.

(f) Every regular sequence on R is a strong parameter sequence.

Proof: Throughout the proof, let $\ell = \ell(\mathbf{x})$. Parts (a) and (b) follow readily from Proposition 2.1 and Remark 2.2. For part (c), we have $f(\mathbf{x})$ is proregular by Remark 2.2. Let Q be a prime of S containing $(f(\mathbf{x}))S$. Then $p = f^{-1}(Q)$ is a prime of R containing $(\mathbf{x})R$. Since the map $R_p \to S_Q$ is faithfully flat, we obtain

$$H_{f(\mathbf{x})}^{\ell}(S)_Q \cong H_{\mathbf{x}}^{\ell}(R)_p \otimes_{R_p} S_Q \neq 0.$$

The converse is proved similarly, again using Remark 2.2.

For part (d), let p be a prime of R minimal over $(\mathbf{x})$. Since $(f(\mathbf{x}))S_p \neq S_p$ and $f(\mathbf{x})$ is a parameter sequence, we have

$$H_{\mathbf{x}}^{\ell}(R)_p \otimes_R S \cong H_{f(\mathbf{x})}^{\ell}(S)_p \neq 0.$$

Hence, $H_{\mathbf{x}}^{\ell}(R)_p \neq 0$.

To prove (e), we first note that $H_i(\mathbf{x}^n) = 0$ for all $i \geq 1$ by Propositions 2.6(b) and 2.7. Hence, $\mathbf{x}$ is weakly proregular. Next, note that p-grade$((\mathbf{x})R_p, R_p) < \infty$ if and only if $(\mathbf{x})R_p \neq R_p$ by Proposition 2.7. Since localization does not decrease p-grade and p-grade (if finite) is bounded above by the length of the sequence, we see that p-grade$((\mathbf{x})R_p) = \ell$ for all primes p containing $(\mathbf{x})R$. Therefore, $H_{(\mathbf{x})}^{\ell}(R)_p \neq 0$ for all $p \supseteq (\mathbf{x})R$ by Proposition 2.7. Part (f) is an immediate consequence of (e).   □

The following lemma allows us to give a simple characterization of parameters on R:

Lemma 3.4. *Let R be a ring and $x \in J(R)$, where $J(R)$ is the Jacobson radical of R. Then $H_x^1(R) = 0$ if and only if x is nilpotent.*

Proof: Suppose $H_x^1(R) \cong R_x/R = 0$. Then there exists an $r \in R$ such that $\frac{1}{x} = \frac{r}{1}$ in R_x. Thus for some i, $(1 - rx)x^i = 0$. As $x \in J(R)$, $1 - rx$ is a unit and hence $x^i = 0$.   □

Corollary 3.5. *Let R be a ring and $x \in R$ a nonunit. Then x is a parameter on R if and only if $\operatorname{ht} xR \geq 1$ and $(0 : x^n) = (0 : x^{n+1})$ for some $n \geq 1$.*

Proof: Let p be a prime minimal over xR. By Lemma 3.4, $H_x^1(R)_p = 0$ if and only if x is nilpotent in R_p, which is the case if and only if $\operatorname{ht} p = 0$. Hence, $H_x^1(R)_p \neq 0$ for all primes p minimal over xR if and only if x is not in any minimal prime of R. As noted in the paragraph preceding Remark 2.2, x is weakly proregular if and only if $(0 : x^n) = (0 : x^{n+1})$ for some n.   □

As a consequence of Proposition 2.4 we have:

Proposition 3.6. *Let R be a ring and $\mathbf{x}$ a parameter sequence on R. Then $\operatorname{ht}(\mathbf{x})R \geq \ell(\mathbf{x})$.*

The following example shows that the converse to this proposition can be false even in the case the sequence in weakly proregular and R is a coherent regular ring:

Example 3.7. Let V be a valuation domain of (Krull) dimension 2. Let m be the maximal ideal of V and P the (unique) prime ideal lying between (0) and m. Choose a non-zero element $x \in P$ and $y \in m \setminus P$. Then x, y is weakly proregular and $\operatorname{ht}(x, y)V = 2$ but x, y is not a parameter sequence.

Proof: Clearly $\operatorname{ht}(x, y)V = \operatorname{ht} m = 2$. Since $(x, y)V = yV$, $H_{x,y}^2(V) = 0$. Thus, x, y is not a parameter sequence. It suffices to show that x, y is weakly proregular. As V is a domain, $H_2(x^n, y^n) = 0$ for all n. Let $\alpha \in H_1(x^{2n}, y^{2n})$ and $(r, s) \in R^2$ a lifting of α. Then $rx^{2n} + sy^{2n} = 0$. The map $H_1(x^{2n}, y^{2n}) \to H_1(x^n, y^n)$ is induced

by the map $(r, s) \rightarrow (rx^n, sy^n)$. As $x \in yV$, $x = by$ for some $\beta \in V$. Let $a = rb^n$. Then

$$rx^n = rb^n y^n = ay^n,$$

$$sy^n = \frac{sy^{2n}}{y^n} = -\frac{rx^{2n}}{y^n} = -\frac{rb^n y^n x^n}{y^n} = -ax^n.$$

Hence, $(rx^n, sy^n) = a(y^n, -x^n)$ is a boundary. Thus, the map $H_1(x^{2n}, y^{2n}) \rightarrow H_1(x^n, y^n)$ is zero. $\square$

We end this section by characterizing strong parameter sequences on trivial extensions of the type described in Proposition 2.9.

Proposition 3.8. *Let the notation be as in the statement of Proposition 2.9. The following are equivalent for a finite sequence $\mathbf{x}$ of S:*

(a) $\mathbf{x}$ is a (strong) parameter sequence on S.
(b) $j(\mathbf{x})$ is a (strong) parameter sequence on R.

Proof: By Proposition 2.9, $\mathbf{x}$ is weakly proregular on S if and only if $j(\mathbf{x})$ is weakly proregular on R. As in the proof of Proposition 2.9, it suffices to prove for sequences $\mathbf{y}$ of R that $\mathbf{y}$ is a parameter sequence on S if and only if it is a parameter sequence on R. Let P be a prime minimal in $\mathrm{Supp}_S S/(\mathbf{y})S$. Since $S_P \cong R_{j(P)} \times (M_i)_{j(P)}$, we may assume $\sqrt{(\mathbf{y})R} = m$. It suffices to show that $H^\ell_{\mathbf{y}}(R) \cong H^\ell_{\mathbf{y}}(S)$ where $\ell = \ell(\mathbf{y}) \geq 1$. From the short exact sequence of R-modules $0 \rightarrow M \rightarrow S \xrightarrow{j} R \rightarrow 0$ we obtain the exact sequence

$$\cdots \rightarrow H^\ell_{\mathbf{y}}(M) \rightarrow H^\ell_{\mathbf{y}}(S) \rightarrow H^\ell_{\mathbf{y}}(R) \rightarrow 0.$$

Since $H^k_{\mathbf{y}}(k(p)) = 0$ for $k \geq 1$ and all $p \in \mathrm{Spec}\, R$, we have $H^\ell_{\mathbf{y}}(M_i) = 0$. The result now follows from the long exact sequence above. $\square$

The following special case of Proposition 3.8 will be needed in the proof of Example 4.3:

Corollary 3.9. *Let (R, m) be a Noetherian local ring and let $S = R \times M_i$ as in Proposition 2.9. Let $\mathbf{x}$ be a finite sequence of elements of S. Then $\mathbf{x}$ is a strong parameter sequence on S if and only if $\mathrm{ht}(x_1, \ldots, x_i)S = i$ for $i = 1, \ldots, \ell(\mathbf{x})$.*

4. THE COHEN-MACAULAY PROPERTY

We begin this section with a definition of Cohen-Macaulay for arbitrary commutative rings:

Definition 4.1. A ring R is called *Cohen-Macaulay* if every strong parameter sequence on R is a regular sequence.

It follows from Remark 3.2 that this definition agrees with the usual definition of Cohen-Macaulay for Noetherian rings (see [BH] or [Mat]). Of course, given the many different characterizations of Noetherian Cohen-Macaulayness, there are many choices for extending the concept to non-Noetherian rings. As we will see

below, the property defined in Definition 4.1 has many similarities to Noetherian Cohen-Macaulayness as well as some stark differences (cf. Examples 4.3 and 4.9). However, we believe the present definition is a good one for exploring homological properties of rings, particularly rings associated in some way to regular rings (e.g., invariant subrings of regular rings).

Below we give equivalent formulations of Cohen-Macaulay in terms of the polynomial grade, Koszul homology, and Čech cohomology of strong parameter sequences:

Proposition 4.2. *Let R be a ring. The following conditions are equivalent:*
(a) R is Cohen-Macaulay.
(b) $\mathrm{grade}(\mathbf{x})R = \ell(\mathbf{x})$ for every strong parameter sequence $\mathbf{x}$ of R.
(c) $\mathrm{p\text{-}grade}(\mathbf{x})R = \ell(\mathbf{x})$ for every strong parameter sequence $\mathbf{x}$ of R.
(d) $H_i(\mathbf{x}; R) = 0$ for all $i \geq 1$ for every strong parameter sequence $\mathbf{x}$ of R.
(e) $H_{\mathbf{x}}^i(R) = 0$ for all $i < \ell(\mathbf{x})$ for every strong parameter sequence $\mathbf{x}$ of R.

Proof: It is clear that (a)$\Rightarrow$(b)$\Rightarrow$(c). By Proposition 2.7, we also have that (c), (d), and (e) are equivalent. It suffices to show (c) $\Rightarrow$ (a). We proceed by induction on $\ell(\mathbf{x})$ to show that all strong parameter sequences on R are regular sequences. If $\mathrm{p\text{-}grade}(x_1)R = 1$ then x_1 is a regular sequence. Suppose all strong parameter sequences of length at most $\ell - 1$ are regular sequences on R. Let $\mathbf{x}$ be a strong parameter sequence of R of length ℓ. Then by the induction hypothesis $\mathbf{x}'$ is a regular sequence on R. Let $R' = R/(\mathbf{x}')R$. By Proposition 2.6(a), $\mathrm{p\text{-}grade}((x_\ell)R', R') = 1$, which implies x_ℓ is a regular element on R'. Hence, $\mathbf{x}$ is a regular sequence on R. $\qquad\square$

The following example shows that, contrary to the Noetherian case, it is not sufficient that $\mathrm{p\text{-}grade}(\mathbf{x})R = \ell(\mathbf{x})$ for all *maximal* strong parameter sequences for a ring R to be Cohen-Macaulay

Example 4.3. *Let (R, m) be a Cohen-Macaulay local ring of dimension $d > 0$ and $S = R \times M_{d-1}$ as in Proposition 2.8. Then $\mathrm{p\text{-}grade}(\mathbf{x})R = \ell(\mathbf{x})$ for all maximal strong parameter sequences of S, but S is not Cohen-Macaulay. In fact, $\mathrm{p\text{-}grade}(\mathbf{x})R = 0$ for all parameter sequences $\mathbf{x}$ of length less than $\dim R$.*

Proof: Combine Proposition 2.9, Corollary 3.9, and Proposition 4.2 $\qquad\square$

We make some elementary observations concerning Cohen-Macaulay rings of small dimension:

Proposition 4.4. *Let R be a ring.*
(a) If $\dim R = 0$ then R is Cohen-Macaulay.
(b) If R is a one-dimensional domain, then R is Cohen-Macaulay.

Proof: Part (a) follows from the fact that a zero-dimensional ring has no parameter sequences. For part (b), note that by Proposition 2.4, the maximum length of a parameter sequence is one. Since every non-zero element of R is a non-zero-divisor, we conclude that R is Cohen-Macaulay. $\qquad\square$

The Cohen-Macaulay property descends along faithfully flat extensions:

Proposition 4.5. *Let $f : R \to S$ be a faithfully flat ring homomorphism. If S is Cohen-Macaulay, then so is R.*

Proof: This follows from Proposition 3.3, part (c) and [Mat, Theorem 7.5]. $\square$

One immediate consequence is:

Corollary 4.6. *Let R be a ring such that the polynomial ring $R[t]$ is Cohen-Macaulay. Then R is Cohen-Macaulay.*

We do not know whether $R[t]$ must be Cohen-Macaulay whenever R is. Likewise, we do not know whether the Cohen-Macaulay property localizes. In both cases, the difficulty lies in linking strong parameter sequences of $R[t]$ (or R_S) to strong parameter sequences of R. It may be that some mild condition on the ring, such as requiring that the sets of minimal primes of finitely generated ideals are finite, is necessary for these properties to hold (cf. [Ma]). However, we do have the following:

Proposition 4.7. *Let R be a ring and suppose R_m is Cohen-Macaulay for all maximal ideals m of R. Then R is Cohen-Macaulay.*

Proof: We proceed by induction on the length of a strong parameter sequence $\mathbf{x}$ on R. If $\ell(\mathbf{x}) = 1$ then x_1 is a regular sequence on R_m for all maximal ideals m containing x_1. Hence x_1 is regular on R. Now assume $\ell = \ell(\mathbf{x}) \geq 2$ and that all strong parameter sequences on R of length $\ell - 1$ are regular sequences. For all maximal ideals m of R containing $(\mathbf{x})$, x_ℓ is regular on $R_m/(\mathbf{x}')R_m$. Hence, x_ℓ is regular on $R/(\mathbf{x}')R$. $\square$

We will call a ring R *locally Cohen-Macaulay* if R_p is Cohen-Macaulay for all $p \in \operatorname{Spec} R$. By Proposition 4.7, if R is locally Cohen-Macaulay then R_S is Cohen-Macaulay for all multiplicatively closed sets S of R. As the following theorem shows, coherent regular rings are locally Cohen-Macaulay. A ring is *regular* if every finitely generated ideal has finite projective dimension ([Be]). A ring is *coherent* if every finitely generated ideal of R is finitely presented. (See [G1] for basic properties of coherent rings.) It is easily seen that every finitely generated ideal of a coherent regular ring has an FFR.

Theorem 4.8. *Let R be a coherent regular ring. Then R is locally Cohen-Macaulay.*

Proof: Since the localization of a coherent regular ring is again coherent regular, it suffices to prove that R is Cohen-Macaulay. Let $\mathbf{x}$ be a strong parameter sequence on R and $I = (\mathbf{x})R$. By induction on $\ell = \ell(\mathbf{x})$, we may assume $\mathbf{x}'$ is a regular sequence on R. We will suppose x_ℓ is a zero-divisor on $R' = R/(\mathbf{x}')R$ and derive a contradiction. Then $x_\ell \in p$ for some $p \in \operatorname{wAss}_R R'$. By Lemma 2.8 we have p-depth $R'_p = 0$. Thus, p-depth $R_p = \ell - 1$. By localizing at p, we can assume (R, m) is a coherent regular quasi-local ring and p-depth $R = \ell - 1$. The Auslander-Buchsbaum formula (Proposition 2.11) yields $\operatorname{pd}_R R/I^t \leq$ p-depth $R = \ell - 1$ for all $t \geq 1$. Therefore, $\operatorname{Ext}_R^\ell(R/I^t, R) = 0$ for all $t \geq 1$. Taking direct limits we obtain $H_I^\ell(R) = 0$. By Proposition 2.3, we have $H_\mathbf{x}^\ell(R) = 0$, contradicting that $\mathbf{x}$ is a parameter sequence on R. $\square$

We note that this theorem answers the question of Glaz ([G2, p. 220]) mentioned in the introduction: Does there exist a definition of Cohen-Macaulay which agrees with the usual notion in the Noetherian case and having the property every coherent regular ring is Cohen-Macaulay?

The following is an example of a two-dimensional Cohen-Macaulay quasi-local ring R with the property that R/xR is not Cohen-Macaulay for some non-zero-divisor x on R.

Example 4.9. Let $S = \mathbb{C}[[x, y]]$ be the ring of formal power series in x and y over the field of complex numbers. Let $R = \mathbb{C} + x\mathbb{C}[[x, y]] \subseteq S$. It is easily seen that R is a quasi-local domain. We prove that:

  (1) R is Cohen-Macaulay.
  (2) R/xyR is not Cohen-Macaulay.

Proof: Let $m = x\mathbb{C}[[x, y]]$ denote the maximal ideal of R. As R is a domain every non-zero element of m is both a parameter and a non-zero-divisor. To prove R is Cohen-Macaulay, it suffices to prove that R has no parameter sequences of length greater than one. In fact, we will show $H_{\mathbf{w}}^i(R) = 0$ for all $i \geq 2$ and for all finite sequences $\mathbf{w}$ of R. Let $\mathbf{w}$ be such a sequence. Clearly, we may assume $(\mathbf{w}) \subseteq m$. Consider the short exact sequence of R-modules

$$0 \to R \to S \to S/R \to 0.$$

Note that as $xS \subseteq R$ we have $m(S/R) = 0$. Hence, $H_{\mathbf{w}}^i(S/R) = 0$ for all $i \geq 1$. Therefore, $H_{\mathbf{w}}^i(R) \cong H_{\mathbf{w}}^i(S)$ for all $i \geq 2$. By the change of rings theorem and since S is Noetherian, we have that $H_{\mathbf{w}}^i(S) \cong H_{(\mathbf{w})S}^i(S)$ for all i. Now, $(\mathbf{w})S \subseteq xS$ and so $\dim S/(\mathbf{w})S > 0$. Since S is a complete local domain of dimension two, we have $H_{(\mathbf{w})S}^i(S) = 0$ for all $i \geq 2$ by the Hartshorne-Lichtenbaum vanishing theorem [BS, Theorem 8.2.1]. Hence, $H_{\mathbf{w}}^i(R) = 0$ for all $i \geq 2$.

Clearly, xy is a non-zero-divisor on R. We claim that x is a parameter on R/xyR. Since $m = \sqrt{xR}$, we need only check that $H_x^1(R/xyR) \neq 0$ and that x is weakly proregular on R/xyR. Clearly x is not nilpotent in R/xyR and so $H_x^1(R/xyR) \neq 0$ by Lemma 3.4. Also, $(xyR :_R x) = xyS = (xyR :_R x^2)$. To see this, first note that $xS \subseteq R$ and $yS \cap R = xyS$. Thus, xyS is a prime ideal of R. Since $xyR \subseteq xyS$ and $x^2 \notin xyS$ we have that $(xyR :_R x^2) \subseteq xyS$. On the other hand, $x(xyS) = xy(xS) \subseteq xyR$ and thus $xyS \subseteq (xyR :_R x) \subseteq (xyR :_R x^2) \subseteq xyS$. Hence, x is weakly proregular on R/xyR. As x is a parameter and a zero-divisor on R/xyR (as $xy^2 \in xyS \setminus xyR$), we see that R/xyR is not Cohen-Macaulay. $\square$

We also note a connection between the present definition of Cohen-Macaulay and the unmixedness notions proposed in [Ha1] and [Ha2] as possible definitions of Cohen-Macaulay for non-Noetherian rings. An ideal I of a ring R is said to be *unmixed* if $\mathrm{wAss}_R R/I = \mathrm{Min}_R R/I$. It is well-known that a Noetherian ring is Cohen-Macaulay if and only if every ideal generated by a parameter sequence is unmixed [Mat, Theorem 17.6]. For arbitrary rings, this unmixedness condition implies the notion of Cohen-Macaulay introduced here, but is properly stronger.

Proposition 4.10. *Let R be a ring such that every ideal generated by a strong parameter sequence is unmixed. Then R is Cohen-Macaulay.*

Proof: Let $\mathbf{x}$ be a strong parameter sequence on R. We use induction on $\ell(\mathbf{x})$ to prove $\mathbf{x}$ is a regular sequence. This is trivial in the case $\ell(\mathbf{x}) = 0$. Suppose $\ell = \ell(\mathbf{x}) > 0$ and that $\mathbf{x}'$ is a regular sequence on R. It suffices to show that x_ℓ is regular on $R/(\mathbf{x}')R$. If x_ℓ is a zero-divisor on $R/(\mathbf{x}')R$ then $x_\ell \in p$ for some $p \in$ $\mathrm{wAss}_R R/(\mathbf{x}')R = \mathrm{Min}_R R/(\mathbf{x}')R$. Thus, $\sqrt{(\mathbf{x})R_p} = \sqrt{(\mathbf{x}')R_p}$. Hence, $H^\ell_{\mathbf{x}}(R)_p = H^\ell_{\mathbf{x}'}(R)_p = 0$, contradicting that $\mathbf{x}$ is a strong parameter sequence on R. Thus, $\mathbf{x}$ is a regular sequence on R and R is Cohen-Macaulay. $\square$

Consequently, the weak Bourbaki unmixed rings and weak Bourbaki height-unmixed rings studied in [Ha1] and [Ha2] are Cohen-Macaulay. However, if R and xy are as in Example 4.9, then R is Cohen-Macaulay, xy is a strong parameter sequence on R, but xyR is not unmixed (as $m \in \mathrm{wAss}_R R/(xy)R \setminus \mathrm{Min}_R R/(xy)R$). Hence the converse of Proposition 4.10 is false.

Let R be an excellent Noetherian local domain of dimension d. The *absolute integral closure* R^+ of R is defined to be the integral closure of R in an algebraic closure of its field of fractions. In [HH2], Hochster and Huneke prove that if char $R = p > 0$ then R^+ is a big Cohen-Macaulay algebra; i.e., every system of parameters for R is a regular sequence on R^+. Using this result, we can show that R^+ is a Cohen-Macaulay ring in the sense introduced here.

Theorem 4.11. *Let R be an excellent Noetherian domain of characteristic $p > 0$. Then R^+ is Cohen-Macaulay.*

Proof: Let $\mathbf{x}$ be a strong parameter sequence on R^+. If we let $S = R[\mathbf{x}]$ then S is also an excellent Noetherian domain and $S^+ = R^+$. Hence, we may assume $\mathbf{x}$ is a sequence of elements in R. By Propositions 4.2 and 2.7, it suffices to prove that $H^i_{\mathbf{x}}(R^+) = 0$ for all $i < \ell(\mathbf{x})$. Since integral closure and Čech cohomology commute with localization, it suffices to prove this in the case when (R, m) is local and $(\mathbf{x})R \subseteq m$. By Proposition 3.3(d), $\mathbf{x}$ is a strong parameter sequence on R. Since R is Noetherian, this means $\mathbf{x}$ is a (partial) system of parameters for R. By [HH2, Theorem 5.15], $\mathbf{x}$ is a regular sequence on R^+. $\square$

As an application of non-Noetherian Cohen-Macaulayness, we consider a conjecture raised by Glaz [G2]: *Let R be a coherent regular ring, G a group of automorphisms of R, and R^G the ring of invariants. Assume that there exists a module retraction $\rho : R \to R^G$ and that R is finitely generated R^G-module. Then R^G is Cohen-Macaulay.* This conjecture is well-known to be true in the case R is Noetherian by the theorem of Hochster and Eagon [HE, Proposition 12]. While we are not able to completely resolve Glaz's conjecture using the present notion of Cohen-Macaulay, we are able to prove it in the case $\dim R = 2$ (Corollary 4.15).

Let $f : R \to S$ be a ring homomorphism. A *module retraction* from S to R is a R-module homomorphism $\rho : S \to R$ such that $\rho(f(r)) = r$ for all $r \in R$. In this case, we call R a *module retract* of S.

We begin with a basic lemma:

Lemma 4.12. *Let $R \subseteq S$ be commutative rings such that R is quasi-local, S is finite over R, and there exists a module retraction $\rho : S \to R$. Then there exists a maximal ideal q of S such that $\rho(xS) = R$ for every $x \in S \setminus q$.*

Proof: Let m be the maximal ideal of R. Since $\rho(mS) \subseteq m$, $\overline{\rho} : S/mS \to R/m$ is a retraction of the extension $R/m \subseteq S/mS$. Thus, it suffices to prove the lemma in the case $R = k$ is a field. Let $q_1 \cap \cdots \cap q_t = 0$ be the primary decomposition for S where $p_i = \sqrt{q_i}$ for each i. (Since k is a field each p_i is a maximal ideal of S.) The Chinese Remainder Theorem gives an isomorphism $S \cong \prod_i Se_i$ where $S/q_i \cong Se_i \subseteq S$ for $i = 1, \ldots, t$. Since $1 = \rho(1) = \rho(e_1) + \cdots + \rho(e_t)$, we have $r_j = \rho(e_j) \neq 0$ for some j. Let $\phi_j : S/q_j \to S$ be defined by $\phi_j(\overline{s}) = se_j$. Setting $\rho_j = r_j^{-1}\rho\,\phi_j$, we obtain a module retraction for the extension $k \subseteq S/q_j$. Suppose the lemma holds for the retraction ρ_j. Then for each $x \in S \setminus p_j$ we have $\rho_j(x(S/q_j)) = k$. Thus $\rho(xS) \supseteq \rho(xe_jS) = r_j\rho_j(x(S/q_j)) = k$. This reduces the proof of the lemma to the case where R is a field and S is local. But this case is trivial, since if x is a unit in S then $\rho(Sx) = \rho(S) = R$. $\square$

Corollary 4.13. *Let R, S, ρ, and q be as in Lemma 4.12. Let M be an R-module and $\mathbf{x}$ a sequence of elements from R. The following hold:*
(a) $M = 0$ if and only if $S_q \otimes_R M = 0$.
(b) If $H^i_{\mathbf{x}}(S \otimes_R M)_q = 0$ then $H^i_{\mathbf{x}}(M) = 0$.
(c) For any ideal I of R,

$$\text{p-grade}(IS_q, S_q \otimes_R M) \leq \text{p-grade}(I, M).$$

(d) If $\mathbf{x}$ is a regular sequence on S_q then $\mathbf{x}$ is a regular sequence on R.

Proof: Observe that the map $\rho' : S \otimes_R M \to M$ given by $\rho'(s \otimes m) = \rho(s)m$ is a retraction of the map $j : M \to S \otimes_R M$ given by $j(m) = 1 \otimes m$ for all $m \in M$. Suppose $S_q \otimes_R M = 0$ and let $m \in M$. Then there exists $t \in S \setminus q$ such that $t \otimes m = t(1 \otimes m) = 0$. By the lemma, there exists $s \in S$ such that $\rho(st) = 1$. Then

$$m = \rho(st)m = \rho'(st \otimes m) = \rho'(0) = 0.$$

To prove (b) we first note that the maps j and ρ induce maps of complexes

$$C(\mathbf{x}; M) \xrightarrow{\psi} S \otimes_R C(\mathbf{x}; M) \xrightarrow{\phi} R \otimes_R C(\mathbf{x}; M) \cong C(\mathbf{x}; M)$$

where ψ is the obvious inclusion induced by j and $\phi = \rho \otimes 1$. This retraction of complexes induces maps on homology $\phi^*_i : H^i_{\mathbf{x}}(S \otimes_R M) \to H^i_{\mathbf{x}}(M)$ where $\phi^*_i(su) = \rho(s)u$ for every $u \in H^i_{\mathbf{x}}(M)$. (The injection ψ allows us to view $H^i_{\mathbf{x}}(M)$ as a direct summand of $H^i_{\mathbf{x}}(S \otimes_R M)$.) Part (b) now follows by the same argument as in the proof of part (a) with ϕ^*_i in place of ρ'. For part (c), we note that we may assume that $I = (\mathbf{x})R$ is a finitely generated ideal by Proposition 2.6(c). The inequality is now immediate from part (b) and Proposition 2.7. Part (d) follows from part (c) and induction on the length of the regular sequence as in the proof of Proposition 4.2. $\square$

We now apply these results to parameter sequences on R:

Theorem 4.14. *Let $R \subseteq S$ be commutative rings such that R is a module retract of S, S is finite over R, and S is a coherent regular ring. Let $\mathbf{x}$ be a strong parameter sequence on R such that $\ell(\mathbf{x}) \leq 2$. Then $\mathbf{x}$ is a regular sequence on R.*

Proof: Let x be a parameter on R and suppose x is a zero-divisor on R. Then $x \in p$ for some $p \in \mathrm{wAss}_R R$. Since the image of x is nonzero in R_p (as x is a parameter), we can localize R and S at p and assume (R, m) is quasi-local. Let q be the maximal ideal of S given by Lemma 4.12. Then the image of x in S_q is nonzero by part (a) of Proposition 4.13. As S_q is a coherent regular quasi-local ring, it is a domain ([Be]). Hence x is a non-zero-divisor on S_q and thus on R as well by Proposition 4.13(c).

Now suppose x, y is a strong parameter sequence on R. By above, we have that x is a non-zero-divisor on R. Suppose y is a zero-divisor on R/xR. Then $y \in p$ for some $p \in \mathrm{wAss}_R R/xR$. Localizing at p, we can assume that (R, m) is quasi-local and p-depth $R/xR = 0$. Again, let q be the maximal ideal of S given by Lemma 4.12. By the argument given in the parameter case, x is a non-zero-divisor on S_q. Since $qS_q = \sqrt{mS_q}$,

$$\begin{aligned}
\text{p-depth}\, S_q/xS_q &= \text{p-grade}(mS_q, S_q/xS_q) \\
&\leq \text{p-grade}(m, R/xR) \qquad \text{(by Corollary 4.13(b))} \\
&= 0.
\end{aligned}$$

Hence, p-depth $S_q = 1$. Therefore, $\mathrm{pd}_{S_q} S_q/(x,y)S_q \leq 1$, which implies that $(x, y)S_q$ is principal. Thus, $H^2_{x,y}(S)_q = 0$ and hence $H^2_{x,y}(R) = 0$ by Proposition 4.13(b), contradicting that x, y is a strong parameter sequence. $\square$

As a special case, we get the following:

Corollary 4.15. *Let R be a coherent regular ring of dimension at most two and G a finite group of automorphisms of R such that the order of G is a unit in R. Let R^G be the subring of invariants of R under the action of G and assume that R is a finite R^G-module. Then R^G is a coherent locally Cohen-Macaulay ring.*

Proof: The averaging map gives a module retraction from R to R^G. Furthermore, $(R^G)_p$ is a module retract of R_p for every prime p of R^G. Since $\dim(R^G)_p = \dim R_p \leq 2$, the maximal length of any parameter sequence on $(R^G)_p$ is two by Proposition 2.4. Applying Theorem 4.14 we see that $(R^G)_p$ is Cohen-Macaulay. Coherence of R^G follows from [G3, Theorem 1]. $\square$

REFERENCES

[A]   Alfonsi, B., *Grade non-Noetherian*, Comm. Alg. 9 (1981), no. 8, 811–840.
[Ba] Barger, S. F., *A theory of grade for commutative rings*, Proc. Amer. Math. Soc. 36 (1972), 365–368.
[Be] Bertin, J., *Anneaux cohérents réguliers*, C. R. Acad. Sci. Paris Sér. A-B 273 (1971), A1–A2.
[Bk] Bourbaki, N., *Commutative Algebra. Chapters 1–7*, Springer-Verlag, Berlin, 1989.

[BS] Brodmann, M. and Sharp, R., *Local Cohomology: an algebraic introduction with geometric applications*, Cambridge Studies in Advanced Mathematics no. **60**, Cambridge, Cambridge University Press, 1998.

[BH] Bruns, W. and Herzog, J., *Cohen-Macaulay Rings*, Cambridge Studies in Advanced Mathematics no. **39**, Cambridge, Cambridge University Press, 1993.

[EN] Eagon, J. A. and Northcott, D. G., *On the Buchsbaum-Eisenbud theory of finite free resolutions*, J. Reine Angew. Math. 262/263 (1973), 205–219.

[G1] Glaz, S., *Commutative Coherent Rings*, Lecture Notes in Mathematics 1371, Springer-Verlag, Berlin, 1989.

[G2] Glaz, S., *Coherence, regularity and homological dimensions of commutative fixed rings* in Commutative Algebra (Ngô Viêt Trung, Aron Simis and Guiseppe Valla, eds.), World Scientific, Singapore, 1992, 89–106.

[G3] Glaz, S., *Fixed rings of coherent regular rings*, Comm. Alg. 20 (1992), 2635–2651.

[G4] Glaz, S., *Homological dimensions of localizations of polynomial rings*, in Zero-dimensional Commutative Rings, John H. Barrett Memorial Lectures and Conference on Commutative Ring Theory (David F. Anderson and David E. Dobbs, eds), Lecture Notes in Pure and Appl. Math., vol. 171, Marcel Dekker, Inc., New York, 1994, 209–222.

[Ha1] Hamilton, T. D., *Weak Bourbaki unmixed rings: A step towards non-noetherian Cohen-Macaulayness*, Rocky Mountain J. Math. 34 (2004), no. 3, 963–977.

[Ha2] Hamilton, T. D., *Unmixedness and the generalized principal ideal theorem*, in Arithmetical Properties of Commutative Rings and Monoids, 282–292, Lect. Notes in Pure Appl. Math., 241, Chapman & Hall/CRC, Boca Raton, FL, 2005.

[Ho] Hochster, M., *Grade-sensitive modules and perfect modules*, Proc. London Math. Soc. (3) 29 (1974), 55–76.

[HE] Hochster, M. and Eagon, J. A., *Cohen-Macaulay rings, invariant theory, and the generic perfection of determinantal loci*, Amer. J. Math. 93 (1971), 1020–1058.

[HH1] Hochster, M. and Huneke, C., *Absolute integral closures are big Cohen-Macaulay algebras in characteristic p*, Bull. Amer. Math. Soc. (New Series), 24 (1991), 137–143.

[HH2] Hochster, M. and Huneke, C., *Infinite integral extensions and big Cohen-Macaulay algebras*, Ann. Math. **135**, 53–89 (1992).

[Mat] Matsumura, H., *Commutative Ring Theory*, Cambridge Studies in Advanced Mathematics no. **8**, Cambridge, Cambridge University Press, 1986.

[Ma] Marley, T., *A note on rings for which finitely generated ideals have only finitely many minimal components*, Comm. Alg., to appear.

[No] Northcott, D.G., *Finite Free Resolutions*, Cambridge Tracts in Mathematics no **71**, Cambridge, Cambridge University Press, 1976.

[Sch] Schenzel, P., *Proregular sequences, local cohomology, and completion*, Math Scand. **92** no. 2, 271–289 (2003).

[V]   Vasconcelos, W. V., *Annihilators of modules with a finite free resolution*, Proc. Amer. Math. Soc., **29** no. 3 (1971), 440–442.

DEPARTMENT OF MATHEMATICS AND STATISTICS, CALIFORNIA STATE UNIVERSITY, SACRAMENTO, SACRAMENTO, CA 95819-6051
    E-mail address: hamilton@csus.edu
    URL: http://www.csus.edu/indiv/h/hamiltont

DEPARTMENT OF MATHEMATICS, UNIVERSITY OF NEBRASKA-LINCOLN, LINCOLN, NE 68588-0130
    E-mail address: tmarley1@math.unl.edu
    URL: http://www.math.unl.edu/~tmarley1